Monetary Economics Notes

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Contents

Chapter 1

New Keynesian Models

Modern New Keynesian models are the standard tool of modern macroeconomic modelling for policy analysis. These models bridge the gap between the methodology of the Real Business Cycle tradition and practical policy evaluation. Introducing price stickiness and imperfect competition in the basic RBC model they provide an useful tool to analyse response of economies to nominal and real shock.

1.1 Readings

Two books provide a complete overview of a very large literature. They are:

- Woodford, Michael (2003): Interest and Prices, Princeton University Press.

- Gali, Jordi (2008) Monetary Policy, Inflation and the Business Cycle, Princeton University Press

The Gali's book provide a very useful discussion of the literature at the end of each chapter. The objective of the following notes is just to clarify some of the maths passages necessary to follow the argument and are not substitute to a careful reading of the Gali book.

1.2 Basic New Keynesian Model

1.2.1 Consumer Problem

$$E_t \sum_{i=0}^{\infty} \beta^i \left[\frac{C_{t+i}^{1-\sigma}}{1-\sigma} - \chi \frac{N_{t+i}^{1+\eta}}{1+\eta} \right] \tag{1.1}$$

$$C_t = \left[\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \tag{1.2}$$

$$P_t = \left[\int_0^1 p_{jt}^{1-\theta} dj \right]^{\frac{1}{1-\theta}} \tag{1.3}$$

$$P_t C_t + B_t = W_t N_t + (1 + r_t) B_{t-1} + \Pi_t \tag{1.4}$$

1) Optimal Allocation of Consumption Expenditure

$$\min_{c_{ij}} \int_0^1 p_{jt} c_{jt} dj \tag{1.5}$$

subject to

$$C_t = \left[\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \tag{1.6}$$

Lagrangian

$$L_t = \int_0^1 p_{jt} c_{jt} dj + \psi \left[C_t - \left[\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \right] \tag{1.7}$$

FOC

$$\frac{\partial L_t}{\partial c_{jt}} = p_{jt} + \psi \left[\left[\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{1}{\theta-1}} c_{jt}^{-\frac{1}{\theta}} \right] = 0 \tag{1.8}$$

Noting that

$$\left[\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{1}{\theta-1}} = C_t^{\frac{1}{\theta}} \tag{1.9}$$

Rewrite (1.8) as

$$p_{jt} - \psi C_t^{\frac{1}{\theta}} c_{jt}^{-\frac{1}{\theta}} = 0 \tag{1.10}$$

$$p_{jt} = \psi C_t^{\frac{1}{\theta}} c_{jt}^{-\frac{1}{\theta}} \tag{1.11}$$

$$p_{jt} c_{jt}^{\frac{1}{\theta}} = \psi C_t^{\frac{1}{\theta}} \tag{1.12}$$

$$c_{jt}^{\frac{1}{\theta}} = \frac{\psi}{p_{jt}} C_t^{\frac{1}{\theta}} \tag{1.13}$$

$$c_{jt} = \left(\frac{\psi}{p_{jt}}\right)^{\theta} C_t \tag{1.14}$$

Or, as usually expressed in the literature

$$c_{jt} = \left(\frac{p_{jt}}{\psi}\right)^{-\theta} C_t \tag{1.15}$$

Substituting (1.15) in (1.2) we get:

$$C_t = \left[\int_0^1 \left(\frac{p_{jt}}{\psi}\right)^{-\theta} C_t^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} = \left(\frac{1}{\psi}\right)^{-\theta} \left[\int_0^1 p_{jt}{}^{1-\theta} dj\right]^{\frac{\theta}{\theta-1}} C_t \tag{1.16}$$

and solving for the Lagrange Multiplier ψ we get

$$1 = \psi^{\theta} \left[\int_0^1 p_{jt}{}^{1-\theta} dj\right]^{\frac{\theta}{\theta-1}} \tag{1.17}$$

$$\psi^{-\theta} = \left[\int_0^1 p_{jt}{}^{1-\theta} dj\right]^{\frac{\theta}{\theta-1}} \tag{1.18}$$

$$\psi = \left[\int_0^1 p_{jt}{}^{1-\theta} dj\right]^{\frac{1}{1-\theta}} \equiv P_t \tag{1.19}$$

Substituting this back in FOC (1.8), we get:

$$c_{jt} = \left(\frac{p_{jt}}{P_t}\right)^{-\theta} C_t \tag{1.20}$$

2) Optimal Dynamic Consumption/Leisure Decision

In real terms, the consumer problem can be written as:

$$E_t \sum_{i=0}^{\infty} \beta^i \left[\frac{C_{t+i}^{1-\sigma}}{1-\sigma} - \chi \frac{N_{t+i}^{1+\eta}}{1+\eta} \right] \tag{1.21}$$

$$C_t + \frac{B_t}{P_t} = \frac{W_t}{P_t} N_t + (1 + r_t) \frac{B_{t-1}}{P_t} + \frac{\Pi_t}{P_t} \tag{1.22}$$

Lagrangian

$$L = E_t \sum_{i=0}^{\infty} \beta^i \left[\frac{C_{t+i}^{1-\sigma}}{1-\sigma} - \chi \frac{N_{t+i}^{1+\eta}}{1+\eta} \right] + \tag{1.23}$$

$$E_t \sum_{i=0}^{\infty} \beta^i \lambda_{t+i} \left[\frac{W_{t+i}}{P_{t+i}} N_{t+i} + (1 + r_{t+i}) \frac{B_{t+i-1}}{P_{t+i}} + \frac{\Pi_{t+i}}{P_{t+i}} - C_{t+i} - \frac{B_{t+i}}{P_{t+i}} \right] \tag{1.24}$$

FOC

$$\frac{\partial L}{\partial C_t} = C_t^{-\sigma} + \lambda_{t+i} = 0 \tag{1.25}$$

$$\frac{\partial L}{\partial N_t} = -\chi N_t^{\eta} + \lambda_t \frac{W_t}{P_t} = 0 \tag{1.26}$$

$$\frac{\partial L}{\partial B_t} = -\lambda_t \frac{1}{P_t} + E_t \left[\beta \lambda_{t+1} (1 + r_{t+1}) \frac{1}{P_{t+i}} \right] = 0 \tag{1.27}$$

Rearranging and using condition (1.25) to eliminate the Lagrange multiplier, we get:

$$\chi N_t^{\eta} = C_t^{-\sigma} \frac{W_t}{P_t} \tag{1.28}$$

$$C_t^{-\sigma} = E_t \left[\beta (1 + r_{t+1}) \left(\frac{P_t}{P_{t+i}} \right) C_{t+1}^{-\sigma} \right] \tag{1.29}$$

As shown before, this implies the following log linear relationships (which we will use later) on

$$w_t - p_t = \sigma c_t + \eta n_t \tag{1.30}$$

$$c_t = E_t \{c_{t+1}\} - \frac{1}{\sigma} \{i_t - E_t \pi_{t=1} - \ln \beta\} \tag{1.31}$$

1.2.2 The Firm

Firms are profit maximisers but they fact three constraints:

- Production Function linear in labour input (the simplest possible - there is no capital - just looking at the short run properties of the model)

$$c_{jt} = Z_t N_{jt} \tag{1.32}$$

- a downward sloping demand curve

$$c_{jt} = \left(\frac{p_{jt}}{P_{t+i}}\right)^{-\theta} C_t \tag{1.33}$$

- nominal inertia like in Calvo (1983).- in each period $(1-\alpha)$ are randomly chosen to set their prices

Real Total Cost, Average Cost, Marginal Cost

$$TC = \frac{W_t}{P_t} N_t = \frac{W_t}{Z_t P_t} c_{jt} \tag{1.34}$$

$$AC = MC = \frac{W_t}{Z_t P_t} \tag{1.35}$$

Productivity shocks affect marginal cost of the firm
Firm pricing problem

$$\max_{p_{jt+i}} \pi = E_t \sum_{i=0}^{\infty} \alpha^i \Delta_{i,t+i} \left[\frac{p_{jt}}{P_{t+i}} c_{jt+i} - \frac{W_{t+j}}{Z_{t+i} P_{t+i}} c_{jt}\right] \tag{1.36}$$

$$= E_t \sum_{i=0}^{\infty} \alpha^i \Delta_{i,t+i} \left[\frac{p_{jt}}{P_{t+j}} \left(\frac{p_{jt}}{P_{t+i}}\right)^{-\theta} C_{t+i} - MC_{t+i} \left(\frac{p_{jt}}{P_{t+i}}\right)^{-\theta} C_{t+i}\right] \tag{1.37}$$

$$= E_t \sum_{i=0}^{\infty} \alpha^i \Delta_{i,t+i} \left[\left(\frac{p_{jt}}{P_{t+i}}\right)^{1-\theta} - MC_{t+i} \left(\frac{p_{jt}}{P_{t+i}}\right)^{-\theta}\right] C_{t+i} \tag{1.38}$$

FOC for the optimal price p_t^*

$$E_t \sum_{i=0}^{\infty} \alpha^i \Delta_{i,t+i} \left[(1-\theta) \left(\frac{1}{p_{jt}} \right) \left(\frac{p_{jt}}{P_{t+i}} \right)^{1-\theta} + \theta MC_{t+i} \left(\frac{1}{p_{jt}} \right) \left(\frac{p_{jt}}{P_{t+i}} \right)^{-\theta} \right] C_{t+i} = 0 \tag{1.39}$$

$$E_t \sum_{i=0}^{\infty} \alpha^i \Delta_{i,t+i} \left[(1-\theta) \left(\frac{p_{jt}}{P_{t+i}} \right) + \theta MC_{t+i} \right] \left(\frac{1}{p_{jt}} \right) \left(\frac{p_{jt}}{P_{t+i}} \right)^{-\theta} C_{t+i} = 0 \tag{1.40}$$

Flexible Price Equlibrium

$$\frac{p_t^*}{P_t} = \frac{\theta}{\theta - 1} MC_t \tag{1.41}$$

$$1 = \frac{\theta}{\theta - 1} \frac{W_t}{Z_t P_t} \tag{1.42}$$

$$\frac{W_t}{P_t} = Z_t \frac{\theta - 1}{\theta} = \frac{\chi N_t^{\eta}}{C_t^{-\sigma}} \tag{1.43}$$

Log linearizing

$$\ln aZ_t = \ln \frac{\chi N_t^{\eta}}{C_t^{-\sigma}} \tag{1.44}$$

taking total derivatives and evaluating at the steady state

$$\frac{1}{\overline{Z}} dZ_t = \eta \frac{1}{\overline{N}} dN_t + \sigma \frac{1}{\overline{C}} dC_t \tag{1.45}$$

define

$$\widehat{x_t^f} = \frac{dX_t}{\overline{X}} \tag{1.46}$$

thus

$$\widehat{z}_t = \eta \widehat{n}_t + \sigma \widehat{c}_t \tag{1.47}$$

Doing the same for the production function

$$\widehat{y^f}_t = \widehat{n}_t + \widehat{z}_t \tag{1.48}$$

Knowing that (without government) in equilibrium consumption equal income

$$\widehat{y^f}_t = \left(\frac{1+\eta}{\sigma+\eta}\right)\widehat{z}_t \tag{1.49}$$

Impulse response function with flexible prices

Sticky Prices Equlibrium

The price index and inflation is determined by the joint solution of the following dynamic equations:
Price index evolves according to:

$$P_t^{1-\theta} = (1-\alpha)\left(p_t^*\right)^{1-\theta} + \alpha P_{t-1}^{1-\theta} \tag{1.50}$$

The expression for the optimal price

$$\frac{p_t^*}{P_t} = \frac{\theta}{\theta-1}\frac{E_t\sum_{i=0}^{\infty}\alpha^i\Delta_{i,t+i}MC_{t+i}\left(\frac{P_t}{P_{t+i}}\right)^{\theta}}{E_t\sum_{i=0}^{\infty}\alpha^i\Delta_{i,t+i}\left(\frac{P_t}{P_{t+i}}\right)^{\theta-1}} \tag{1.51}$$

Log linearising the price index, we get:

$$\widehat{P}_t = (1-\alpha)\,\widehat{p}_t^* + \alpha\widehat{P}_{t-1} \tag{1.52}$$

Log linearising the second we get

$$\widehat{p}_t^* = E_t\sum_{i=0}^{\infty}\alpha^i\beta^i\left(\widehat{MC}_{t+i} + \widehat{P}_{t+i}\right) \tag{1.53}$$

This can be quasi-differentiated to get:

$$\widehat{p}_t^* = \alpha\beta\widehat{p}_{t+1}^* + \widehat{MC}_t + \widehat{P}_t \tag{1.54}$$

Combining the two equations gives:

$$\frac{\widehat{P}_t}{1-\alpha} - \frac{\alpha}{1-\alpha}\widehat{P}_{t-1} = \alpha\beta\left(\frac{\widehat{P}_{t+1}}{1-\alpha}\right) - \alpha\beta\frac{\alpha}{1-\alpha}\widehat{P}_t + \widehat{MC}_t + \widehat{P}_t \tag{1.55}$$

Solving for inflation, yields:

$$\pi_t = \beta E_t \pi_{t+1} + \widetilde{\kappa} \widehat{MC}_t \tag{1.56}$$

where

$$\widetilde{\kappa} = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} \tag{1.57}$$

Which is the New Keynesian Phillips.
Express the NKPC in term of deviation from the flexible price equilibrium
Recall the marginal cost is given by:

$$MC = \frac{W_t}{Z_t P_t} \tag{1.58}$$

Log linearizing, we get:

$$\widehat{MC}_t = \widehat{W}_t - \widehat{P}_t - \widehat{Z}_t \tag{1.59}$$

Usind the labour supply condition, $\widehat{y}_t = \widehat{n}_t + \widehat{z}_t$

$$\begin{aligned}
\widehat{MC}_t &= \widehat{W}_t - \widehat{P}_t - (\widehat{y}_t - \widehat{n}_t) && (1.60)\\
&= (\sigma+\eta)\left[\widehat{y}_t - \left(\frac{1+\eta}{\sigma+\eta}\right)\widehat{z}_t\right] && (1.61)\\
&= (\sigma+\eta)\left[\widehat{y}_t - \widehat{y}_t^f\right] && (1.62)
\end{aligned}$$

Which makes possible to write the NKPC in term of deviation from the flexible price equilibrium

$$\pi_t = \beta E_t \pi_{t+1} + \kappa\left(\widehat{y}_t - \widehat{y}_t^f\right) \tag{1.63}$$

where $\kappa = (\sigma+\eta)\,\widetilde{\kappa}$

Chapter 2

Optimal Monetary Policy

2.0.3 Optimal Policy Problem

$$\min_{y} L_t = E_t \sum_{\tau=0}^{\infty} \frac{1}{2} \beta^{\tau} \left\{ \left(\pi_{t+\tau} - \pi^T \right)^2 + \gamma y_{t+i}^2 \right\} \tag{2.1}$$

subject to

$$\pi_t = \beta E_t \pi_{t+1} + k y_t + \varepsilon_t \tag{2.2}$$

$$y_t = E_t y_{t+1} - \gamma \left(i_t - E_t \pi_{t+1} \right) + \xi_t \tag{2.3}$$

Lagrangian

$$\min_{i} L_t = E_t \sum_{\tau=0}^{\infty} \beta^{\tau} \left\{ \begin{array}{c} \left[\frac{1}{2} \left(\pi_{t+\tau}^2 + \gamma y_{t+i}^2 \right) \right] + \\ \lambda_{t+\tau} \left[\pi_{t+\tau} - \beta E_t \pi_{t+1+\tau} - k y_{t+\tau} - \varepsilon_{t+\tau} \right] \\ + \mu_{t+\tau} \left[y_{t+\tau} - E_t y_{t+1+\tau} + \gamma \left(i_{t+\tau} - E_t \pi_{t+1+\tau} \right) - \xi_{t+\tau} \right] \end{array} \right\} \tag{2.4}$$

FOC respect to $i_{t+\tau}$ is equal to

$$\mu_{t+\tau} \gamma = 0$$

- Problem can be stated just in term of π and y

2.0.4 Discretionary Solution

Period by period problem

$$\min_{y} L = \frac{1}{2} E\left[\pi_t^2 + \gamma y_t^2\right] \tag{2.5}$$

subject to

$$\pi_t = \beta E_t \pi_{t+1} + k y_t + \varepsilon_t \tag{2.6}$$

$$\varepsilon_{t+1} = \rho \varepsilon_t + v_t, \qquad 0 < \rho < 1, \qquad v_t \text{ is } iid$$

FOC

$$\frac{dL}{dy_t} = k\pi_t + \gamma y_t = 0 \tag{2.7}$$

$$y_t = -\frac{k}{\gamma}\pi_t \tag{2.8}$$

Solution

$$\pi_t = \frac{\gamma}{\gamma + k^2}\beta E_t \pi_{t+1} + \frac{\gamma}{\gamma + k^2}\varepsilon_t. \tag{2.9}$$

Solving forward

$$\pi_t = \frac{\gamma}{k^2 + \gamma(1 - \beta\rho)}\varepsilon_t \tag{2.10}$$

2.0.5 Commitment - Timeless Perspective

$$\min_{i} L_t = E_t \sum_{\tau=0}^{\infty} \frac{1}{2}\beta^{\tau}\left(\pi_{t+\tau}^2 + \gamma y_{t+i}^2\right) \tag{2.11}$$

subject to

$$\pi_t = \beta E_t \pi_{t+1} + k y_t + \varepsilon_t$$

where, as before, $\varepsilon_{t+1} = \rho \varepsilon_t + v_t$ is the stochastic policy process
FOC

$$\min_i L_t = E_t \sum_{\tau=0}^{\infty} \beta^{\tau} \left\{ \left[\frac{1}{2} \left(\pi_{t+\tau}^2 + \gamma y_{t+i}^2 \right) \right] + \lambda_{t+\tau} \left[\pi_{t+\tau} - \beta E_t \pi_{t+1+\tau} - k y_{t+\tau} - \varepsilon_{t+\tau} \right] \right\} \tag{2.12}$$

FOC with respect to π_t and y_t for $\tau = 0$

$$\frac{\delta \pounds}{\delta \pi_t} = \pi_t + \lambda_t = 0 \tag{2.13}$$

$$\frac{\delta \pounds}{\delta y_t} = \gamma y_t - k\lambda_t = 0 \tag{2.14}$$

and for $\tau > 0$

$$\frac{\delta \pounds}{\delta \pi_{t+\tau}} = \beta^{\tau} \left(\pi_{t+\tau} + \lambda_{t+\tau} - \lambda_{t+\tau-1} \right) = 0 \tag{2.15}$$

$$\frac{\delta \pounds}{\delta y_{t+\tau}} = \gamma y_{t+\tau} - k\lambda_t = 0 \tag{2.16}$$

FOC (2)

$$for.\tau \geq 0 \rightarrow \frac{\delta \pounds}{\delta y_{t+\tau}} = \gamma y_{t+\tau} - k\lambda_t = 0 \tag{2.17}$$

$$for.\tau = 0 \rightarrow \frac{\delta \pounds}{\delta \pi_t} = \pi_t + \lambda_t = 0 \tag{2.18}$$

$$for.\tau > 0 \rightarrow \frac{\delta \pounds}{\delta \pi_{t+\tau}} = \pi_{t+\tau} + \lambda_{t+\tau} - \lambda_{t+\tau-1} = 0 \tag{2.19}$$

2.0.6 Policy Inertia

Combining (2.19) and (2.17) we get

$$\pi_t + \frac{\gamma}{k} \left(y_t - y_{t-1} \right) = 0$$

which can be rewritten as

$$y_t = y_{t-1} - \frac{k}{\gamma} \pi_t \tag{2.20}$$

Solution
substituting in the supply equation

$$\begin{aligned} -\frac{\gamma}{k}\left(y_t - y_{t-1}\right) &= -\beta\frac{\gamma}{k}\left(E_t y_{t+1} - y_t\right) + k y_t + \varepsilon_t \\ -\frac{\gamma}{k}y_t - \beta\frac{\gamma}{k}y_t - k y_t &= -\beta\frac{\gamma}{k}E_t y_{t+1} - \frac{\gamma}{k}y_{t-1} + \varepsilon_t \\ \left(1+\beta+\frac{k^2}{\gamma}\right)y_t &= \beta E_t y_{t+1} + y_{t-1} - \frac{k}{\gamma}\varepsilon_t \end{aligned} \tag{2.21}$$

Second order difference equation to solve with **undetermined coefficients method**
Undetermined Coefficients Method
First posit a solution for y_t that is a function of the state $(y_{t-1,}\varepsilon_t)$:

$$y_t = a y_{t-1} + b\varepsilon_t \tag{2.22}$$

This, togheter with the assumption of shocks following a AR(1) process, gives

$$E_t y_{t+1} = a y_t + b\rho\varepsilon_t = a^2 y_{t-1} + b\left(a+\rho\right)\varepsilon_t \tag{2.23}$$

substituting in (2.21) we get

$$\left(1+\beta+\frac{k^2}{\gamma}\right)\left(a y_{t-1} + b\varepsilon_t\right) = \beta\left[a^2 y_{t-1} + b\left(a+\rho\right)\varepsilon_t\right] + y_{t-1} - \frac{k}{\gamma}\varepsilon_t \tag{2.24}$$

that is:

$$\left(1+\beta+\frac{k^2}{\gamma}\right)\left(a y_{t-1} + b\varepsilon_t\right) = \left(1-\beta a^2\right) y_{t-1} + \left(\beta b\left(a+\rho\right) - \frac{k}{\gamma}\right)\varepsilon_t \tag{2.25}$$

Find the value of a and b that match the coefficients:

$$\left(1+\beta+\frac{k^2}{\gamma}\right)a = \left(1-\beta a^2\right)_t \tag{2.26}$$

$$\left(1+\beta+\frac{k^2}{\gamma}\right)b = \beta b\left(a+\rho\right) - \frac{k}{\gamma} \tag{2.27}$$

Equation for a quadratic - choose the root $|a| < 1$. The solution for b is instead unique, i.e

$$b = -\left(\frac{k}{\gamma\left(1+\beta\left(1-a-\rho\right)\right)+k^2}\right) \tag{2.28}$$

Decision rule for π_t is:

$$\pi_t = \frac{\gamma}{k}\left(y_t - y_{t-1}\right) \tag{2.29}$$

$$\pi_t = -\frac{\gamma}{k}\left(ay_{t-1} - \left(\frac{k}{\gamma\left(1+\beta\left(1-a-\rho\right)\right)+k^2}\right)\varepsilon_t - y_{t-1}\right) \tag{2.30}$$

$$\pi_t = \frac{\gamma}{k}\left(1-a\right)y_{t-1} + \left(\frac{\gamma}{\gamma\left(1+\beta\left(1-a-\rho\right)\right)+k^2}\right)\varepsilon_t \tag{2.31}$$

- Under both precommitment and discretion, monetary policy completely offsets the impacts of the demand shock, ξ_t, so ξ_t does not affect either y_t or π.

- Cost shocks, ε_t, have different impacts under precommitment and discretion because under discretion there is a stabilization bias.

- There is history dependence of optimal policy under precommitment, but none under discretion. The history dependence is a mean by which commitment is implemented.