Introduction to VAR Models

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Introduction

Origins of VAR models

Sims "Macroeconomics and Reality" Econometrica 1980

It should be feasible to estimate large macromodels as unrestricted reduced forms, treating all variables as endogenous

- Natural extension of the univariate autoregressive model to multivariate time series
- Especially useful for describing the dynamic behaviour of economic and financial time series
- Benchmark in forecasting
- Used for structural inference

Consider a bivariate $Y_t=(y_t, z_t)$, first-order VAR model:

$$y_t = b_{10} - b_{12} z_t + \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + \varepsilon_{yt}$$

$$z_t = b_{20} - b_{21} y_t + \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + \varepsilon_{zt}$$

The two variables y and z are endogenous.

The error terms (structural shocks) ε_{yt} and ε_{zt} are white noise innovations with standard deviations σ_y and σ_z and a **zero** covariance.

Shock ε_{yt} affects y directly and z indirectly.

There are 10 (8 coefficients and two standard deviations of the errors) parameters to estimate.

- The structural is not estimable directly
- VAR in reduced form is estimable.
- In a reduced form representation y and z are just functions of lagged y and z.
- To solve for a reduced form write the structural VAR in matrix form as:

$$\begin{bmatrix} 1 & b_{12} \\ b_{21} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} b_{10} \\ b_{20} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{zt} \end{bmatrix}$$

or, in short

$$BY_t = \Gamma_0 + \Gamma_1 Y_{t-1} + \varepsilon_t$$

- Premultipication by B^{-1} allow us to obtain a standard VAR(1):

$$BY_t = \Gamma_0 + \Gamma_1 Y_{t-1} + \varepsilon_t$$
$$Y_t = B^{-1}\Gamma_0 + B^{-1}\Gamma_1 Y_{t-1} + B^{-1}\varepsilon_t$$
$$Y_t = A_0 + A_1 Y_{t-1} + a_t$$

- This reduced form can be estimated (by OLS equation by equation)
- Before estimating
  - Determine the optimal lag length of the VAR
  - Determine stability conditions (roots of the system inside the unit circle)
- After estimating the reduced form
  - Hypothesis Testing – Granger Casuality
  - Impulse Response Function
  - Variance Decomposition
  - Identification of Structural VAR

Information Criteria (IC) can be used to choose the "right" number of lags in a VAR(p): that minimizes IC(p) for p=1, ..., P .

$$AIC = \ln\left|\Sigma(p)\right| + \frac{2}{T}(n^2 p + n)$$

$$SBC = \ln\left|\Sigma(p)\right| + (n^2 p + n)\frac{\ln(T)}{T}$$

AIC criterion asymptotically overestimates the order with positive probability

SBC criterion estimates the order consistently if the true p is less than the p(max)

Same concept of stability than in univariate time series

The system is stable if the root of the matrix A_1 of the system are all less than 1 in absolute value

$$Y_t = A_0 + A_1 Y_{t-1} + a_t$$

If not Vector Error Correction Models and Cointegration Models more appropriate.

Granger (1969) “Investigating Causal Relations by Econometric Models and Cross-Spectral Methods”, *Econometrica, 37*

Consider two random variables X_t, Y_t

Forecast of X_t, s periods ahead

$$\hat{X}_t(s)^{(1)} = E(X_{t+s} \mid X_t, X_{t-1}, \ldots.) \quad \hat{X}_t(s)^{(2)} = E(X_{t+s} \mid X_t, X_{t-1}, \ldots Y_t, Y_{t-1}, \ldots.)$$

Define Minimum square error $MSE(\hat{X}_t(s)) = E(X_{t+s} - \hat{X}_t(s))^2$

If $MSE(\hat{X}_t(s)^{(1)}) = MSE(\hat{X}_t(s)^{(2)})$

then Y_t does not Granger-cause X_t $\forall s > 0$

$\Leftrightarrow X_t$ is exogenous with respect to Y_t

$\Leftrightarrow Y_t$ is not linearly informative to forecast X_t

Granger Causality Test

Assume a lag length of p

$$X_t = c_1 + \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + ...\alpha_p X_{t-p} + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + ...\beta_p X_{t-p} + a_t$$

Estimate by OLS and test for the following hypothesis

$$H_0 : \beta_1 = \beta_2 = ... = \beta_p = 0 \;\; (Y_t \text{ does not Granger - cause } X_t)$$

$$H_1 : \text{any } \beta_i \neq 0$$

Unrestricted sum of squared residuals $RSS_1 = \sum_t \hat{a}_t^{\,2}$

Restricted sum of squared residuals $RSS_2 = \sum_t \hat{\hat{a}}_t^{\,2}$

$$F = \frac{(RSS_2 - RSS_1)/p}{RSS_1/(T-2p-1)}$$

reject if $F > F_{\alpha,(p,T-2p-1)}$

Important – Stationarity of the data

Objective: Show the reaction of the system to a shock

$$Y_t = c + \Phi_1 Y_{t-1} + \Phi_2 Y_{t-2} + ... + \Phi_p Y_{t-p} + a_t$$

If the system is covariance - stationary,

$$Y_t = \mu + \Psi(L) a_t = \mu + a_t + \Psi_1 a_{t-1} + \Psi_2 a_{t-2} + ...$$

$$\Psi(L) = [\Phi(L)]^{-1}$$

Redating at time $t+s$:

$$Y_{t+s} = \mu + a_{t+s} + \Psi_1 a_{t+s-1} + \Psi_2 a_{t+s-2} + ... + \Psi_s a_t + \Psi_{s+1} a_{t-1} + ...$$

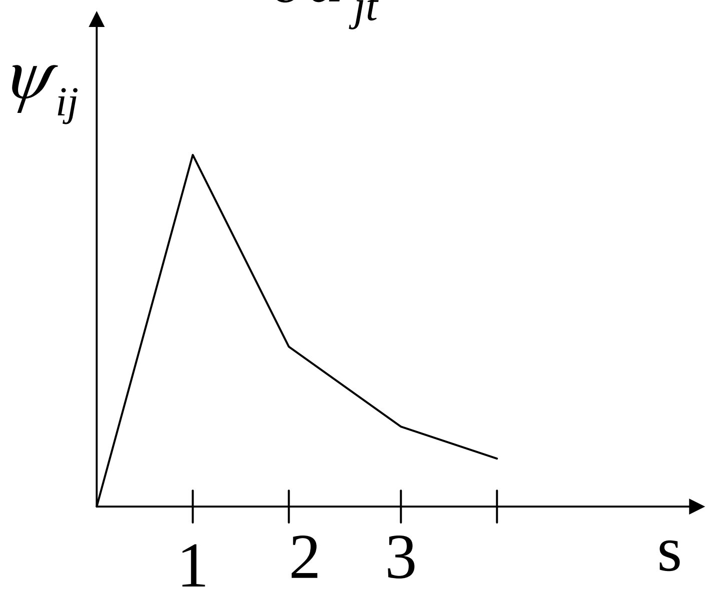
$$\frac{\partial Y_{t+s}}{\partial a'_t} = \Psi_s = \left[\psi_{ij}^{(s)}\right]_{n \times n} \qquad \text{(multipliers)}$$

$$\frac{\partial y_{i,t+s}}{\partial a_{jt}} = \psi_{ij}^{(s)} \longrightarrow$$

Reaction of the i-variable to a unit change in innovation j

Impulse-response function: response of $y_{i,t+s}$ to one-time impulse in y_{jt} with all other variables dated t or earlier held constant.

$$\frac{\partial y_{i,t+s}}{\partial a_{jt}} = \psi_{ij}$$



Example: IRF for a VAR(1)

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{1t-1} \\ y_{2_{t-1}} \end{bmatrix} + \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix}; \quad \Sigma_a = \begin{bmatrix} \sigma_1{}^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2{}^2 \end{bmatrix}$$

$$t < 0 \quad y_{1t} = y_{2t} = 0$$

$$t = 0 \quad a_{20} = 1 \quad (y_{2t} \text{ increases by 1 unit})$$

(no more shocks occur)

Reaction of the system

$$\begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (\textit{impulse})$$

$$\begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \phi_{11} \\ \phi_{22} \end{bmatrix}$$

$$\begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} y_{1s} \\ y_{2s} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}^s \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \Phi_1{}^s \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Contribution of the j-th orthogonalized innovation to the MSE of the s-period ahead forecast

If shocks does not explain none of the forecast error variance of at all forecast horizon we can say that the sequence is exogenous

- Is it possible to recover the parameters in the structural VAR from the estimated parameters in the standard VAR? No!!
- There are 10 parameters in the bivariate structural VAR(1) and only 9 estimated parameters in the standard VAR(1).
- The VAR is underidentified.
- If one parameter in the structural VAR is restricted the standard VAR is exactly identified.
- Sims (1980) suggests a recursive system to identify the model letting $b_{21}=0$. **Choleski decomposition**.

$$\begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} b_{10} \\ b_{20} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{zt} \end{bmatrix}$$

- **b_{21}=0** implies

$$\begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} 1 & -b_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_{10} \\ b_{20} \end{bmatrix} + \begin{bmatrix} 1 & -b_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & -b_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{zt} \end{bmatrix}$$

$$\begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} a_{10} \\ a_{20} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix}$$

The parameters of the structural VAR can now be identified from the following 9 equations

$$a_{10} = b_{10} - b_{12} b_{20} \quad a_{20} = b_{20} \quad \mathrm{var}(e_1) = \sigma_y^2 + b_{12}^2 \sigma_z^2$$

$$a_{11} = \gamma_{11} - b_{12} \gamma_{21} \quad a_{21} = \gamma_{21} \quad \mathrm{var}(e_2) = \sigma_z^2$$

$$a_{12} = \gamma_{12} - b_{12} \gamma_{22} \quad a_{22} = \gamma_{22} \quad \mathrm{cov}(e_1, e_2) = -b_{12} \sigma_z^2$$

- Both structural shocks can now be identified
- b_{21}=0 implies y does not have a contemporaneous effect on z.
- Both ε_{yt} and ε_{zt} affect y contemporaneously but only ε_{zt} affects z contemporaneously.
- The residuals of e_{2t} are due to pure shocks to z.
- There are other methods used to identify models – Restrictions coming from theory – (Sims Bernake, Blanchard and Quah etc)

Critics on VAR

- A VAR model can be a good forecasting model, but it is an atheoretical model (as all the reduced form models are).
- To calculate the IRF, the order matters: Identification not unique.
- Sensitive to the lag selection
- Dimensionality problem.

Standard Tool for Macroeconomic Analysis

An example of VAR analysis

Leeper Sims and Zha (1996)
"What does monetary policy do"

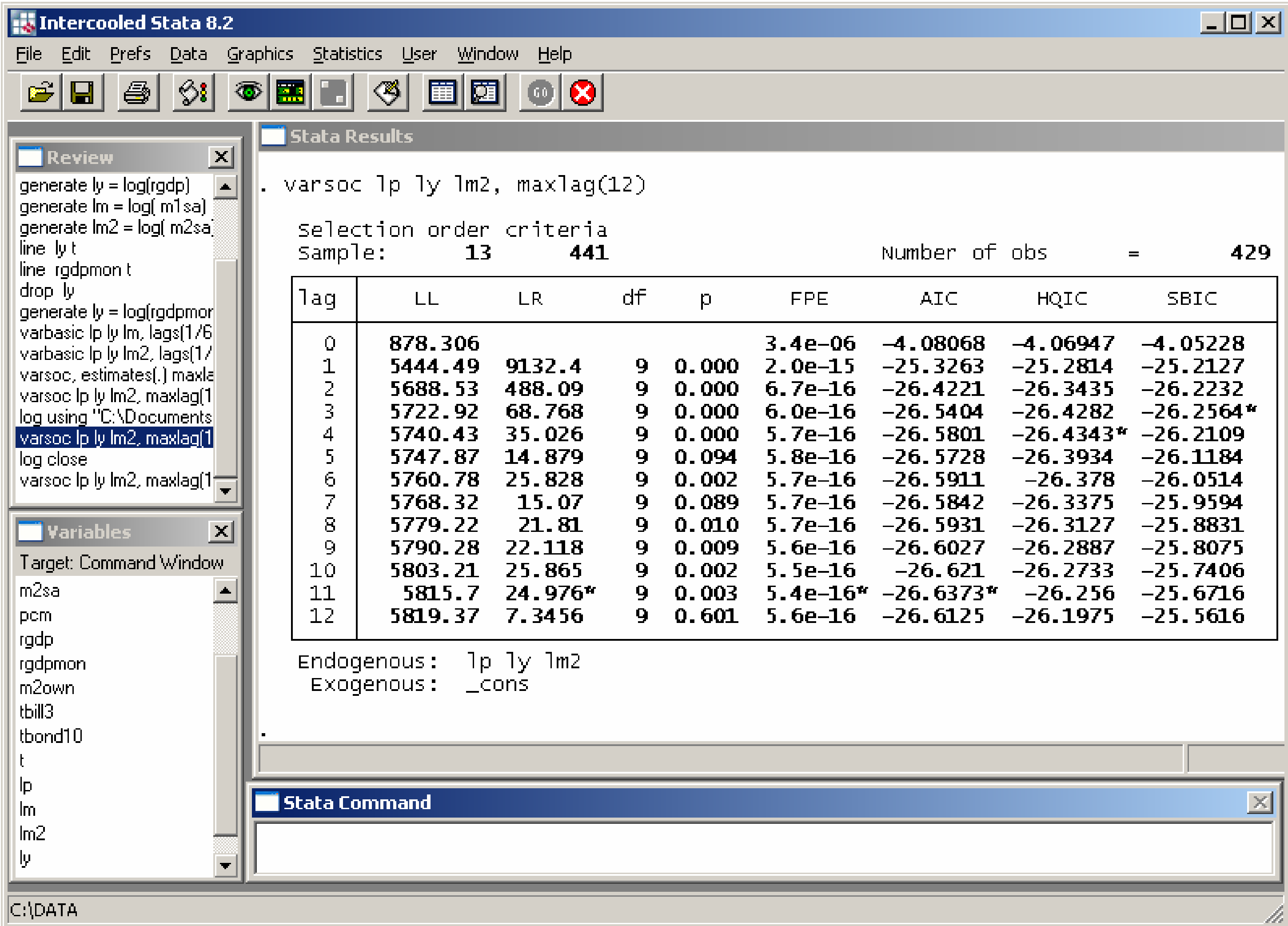
The model

$$P_t = \text{Prices}, X_t = \text{Income}, M_t = \text{Money}$$

$$Y_t = \begin{bmatrix} P_t \\ X_t \\ M_t \end{bmatrix} \quad Y_t = c + \Phi_1 \begin{bmatrix} P_{t-1} \\ X_{t-1} \\ M_{t-1} \end{bmatrix} + \Phi_2 \begin{bmatrix} P_{t-2} \\ X_{t-2} \\ M_{t-2} \end{bmatrix} + ..\Phi_p \begin{bmatrix} P_{t-p} \\ X_{t-p} \\ M_{t-p} \end{bmatrix} + a_t$$

Where $\Phi_1 = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix}^{(1)}$ And $E(a_t) = 0 \quad E(a_t a_\tau') = \begin{cases} \Omega & t = \tau \\ 0 & t \neq \tau \end{cases}$

Steps:
- Optimal Lag Length
- Stability
- Residual Analysis
- Identification (if interest in structural form or in structural shocks)
- Impulse Response Function
- Variance Decomposition



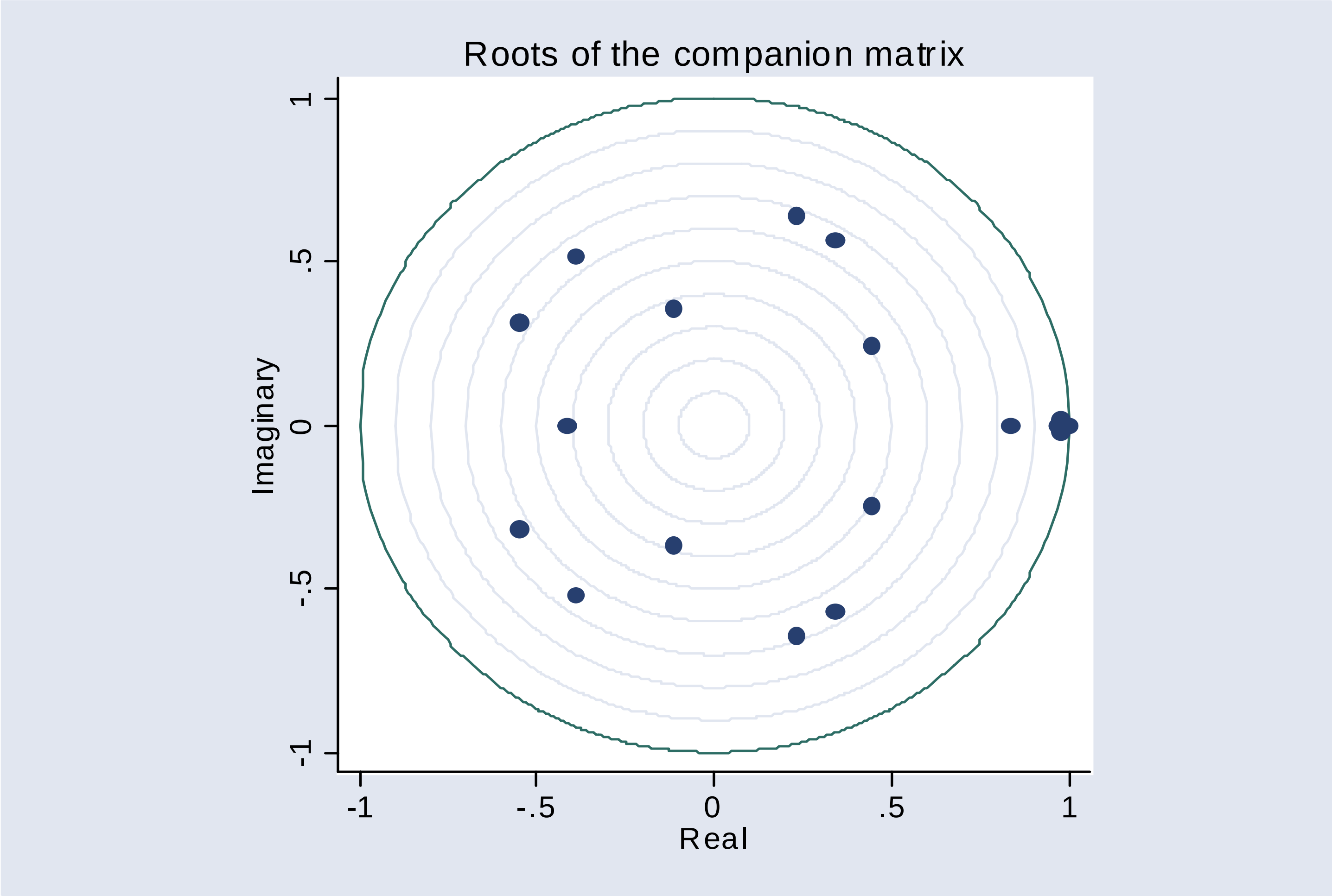

```
. varsoc lp ly lm2, maxlag(12)

  Selection order criteria
  Sample:       13       441                     Number of obs      =       429
```

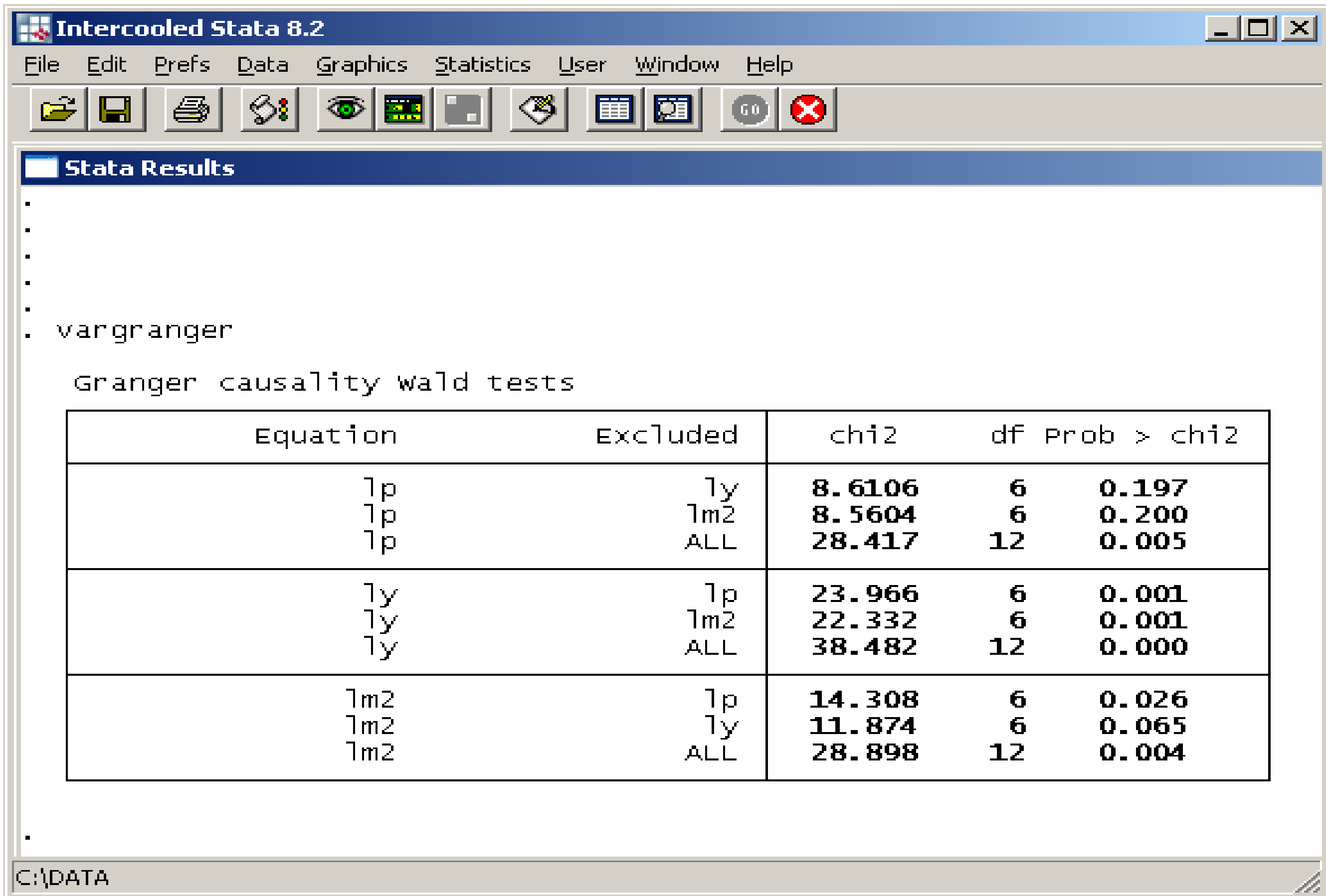
lag	LL	LR	df	p	FPE	AIC	HQIC	SBIC
0	878.306				3.4e-06	-4.08068	-4.06947	-4.05228
1	5444.49	9132.4	9	0.000	2.0e-15	-25.3263	-25.2814	-25.2127
2	5688.53	488.09	9	0.000	6.7e-16	-26.4221	-26.3435	-26.2232
3	5722.92	68.768	9	0.000	6.0e-16	-26.5404	-26.4282	-26.2564*
4	5740.43	35.026	9	0.000	5.7e-16	-26.5801	-26.4343*	-26.2109
5	5747.87	14.879	9	0.094	5.8e-16	-26.5728	-26.3934	-26.1184
6	5760.78	25.828	9	0.002	5.7e-16	-26.5911	-26.378	-26.0514
7	5768.32	15.07	9	0.089	5.7e-16	-26.5842	-26.3375	-25.9594
8	5779.22	21.81	9	0.010	5.7e-16	-26.5931	-26.3127	-25.8831
9	5790.28	22.118	9	0.009	5.6e-16	-26.6027	-26.2887	-25.8075
10	5803.21	25.865	9	0.002	5.5e-16	-26.621	-26.2733	-25.7406
11	5815.7	24.976*	9	0.003	5.4e-16*	-26.6373*	-26.256	-25.6716
12	5819.37	7.3456	9	0.601	5.6e-16	-26.6125	-26.1975	-25.5616

```
  Endogenous:  lp ly lm2
   Exogenous:  _cons
.
```

Stability of the VAR – Roots of the companion matrix



Command **varstable, graph** after **var** or **svar**



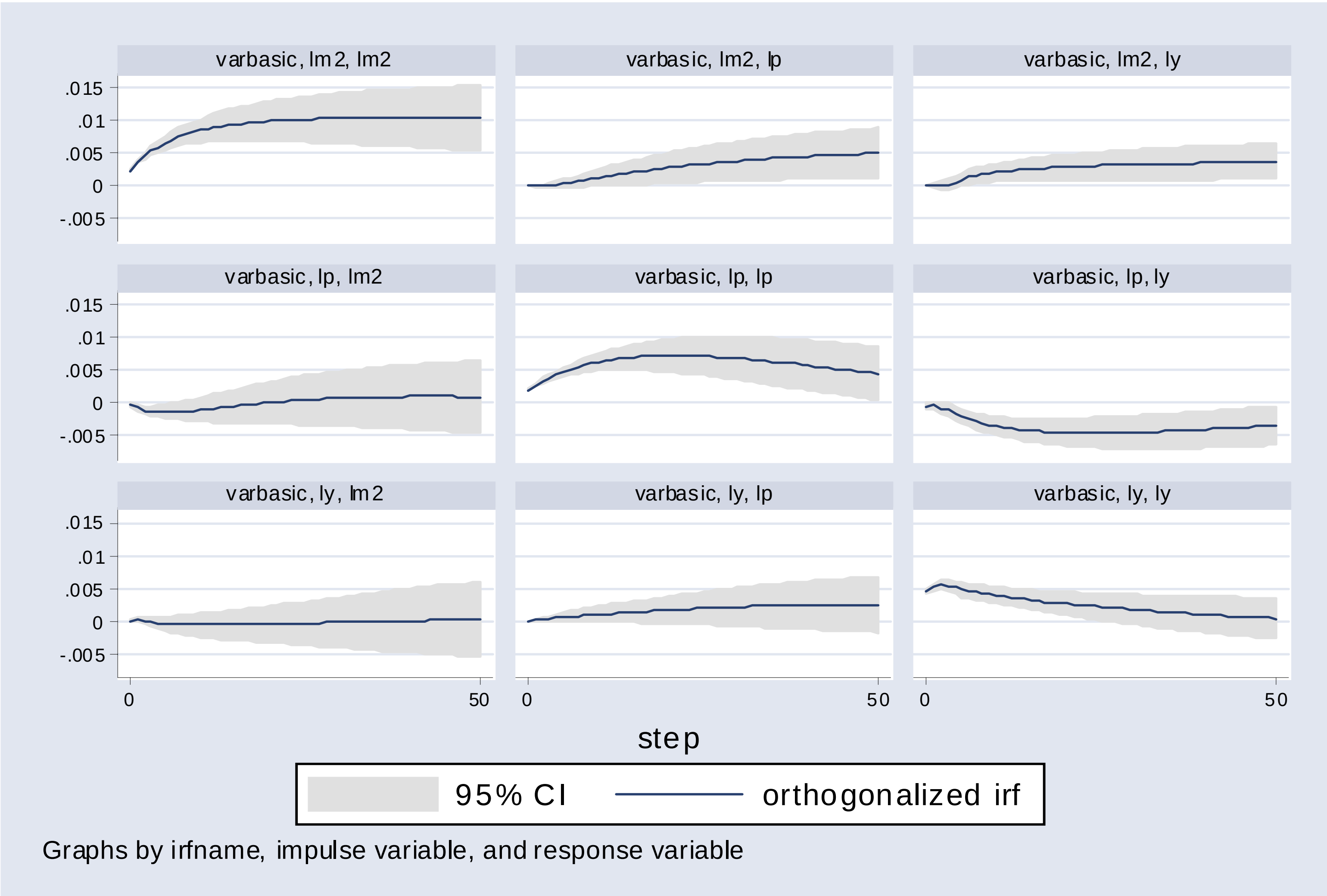
. vargranger

Granger causality Wald tests

Equation	Excluded	chi2	df	Prob > chi2
lp	ly	8.6106	6	0.197
lp	lm2	8.5604	6	0.200
lp	ALL	28.417	12	0.005
ly	lp	23.966	6	0.001
ly	lm2	22.332	6	0.001
ly	ALL	38.482	12	0.000
lm2	lp	14.308	6	0.026
lm2	ly	11.874	6	0.065
lm2	ALL	28.898	12	0.004

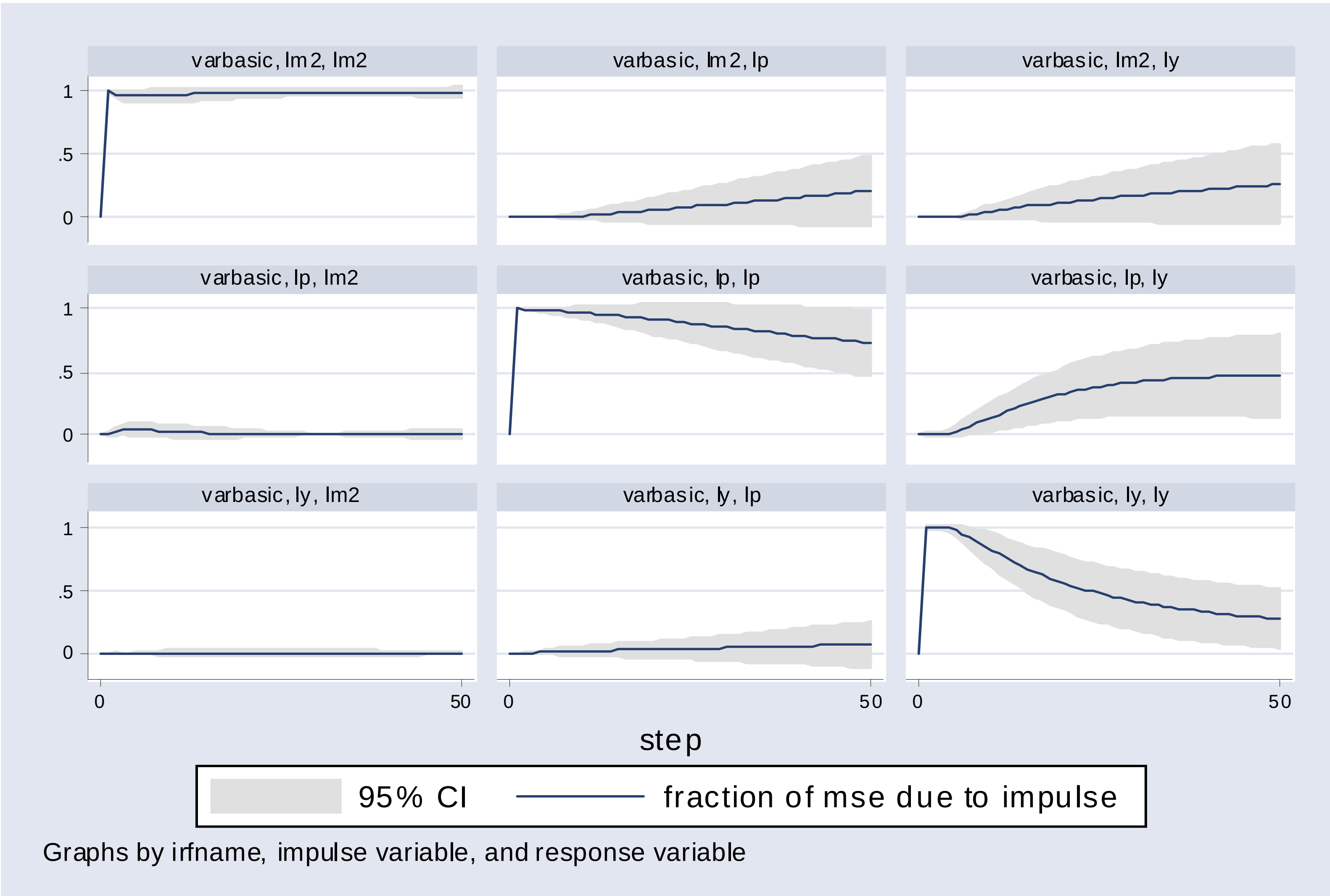
Command – vargranger after varbasic, var or svar

Cholesky Decomposition – Order Prices/Income/Money



varbasic lp ly lm2, lags(1/6) step(50) oirf

Forecast Error Variance Decomposition - FEVD



varbasic lp ly lm2, lags(1/6) step(50) fevd