The Barro-Gordon Model - Detailed Derivation

Nicola Viegi
University of Pretoria

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This note gives a step-by-step derivation of the Barro-Gordon model. The model is interesting because it illustrates in the simplest possible way the relationship between private sector expectations and the functioning of monetary policy. The results are more general than the basic set up presented here. The model suggests that to evaluate the effect of policies the policy-maker has to consider the private sector response to the policy itself. This is true for monetary policy but it also true for fiscal policy or any other policy, for that matter.
The model presents a very simplified version of the economy, in which the relationship between output and inflation is summarized in the following Phillips Curve relationship:

$$y = y^* + b\left(\pi - \pi^e\right) \tag{1}$$

where y is output, $y*$ is equlibrium output, π is actual inflation, π^e is private sector expected inflation formed at the beginning of the period. To rationalize the mechanism behind (1), we can immagine a situation where workers fix their wages at the beginning of the period equal to the expected inflation and the central bank fixes inflation. If actual inflation is higher than expected inflation than prices grow faster than wages and firms make more profits thus expanding their production, until wages are adjusted in the following period.
The Central Bank decides the optimal inflation rate by following a standard quadratic loss function in differences between inflation and inflation target and between output and output target.

$$\begin{aligned} L &= \frac{1}{2}\left[a\left(\pi - \pi^*\right)^2 + \left(y - ky^*\right)^2\right] \\ a &> 0, k > 1, \pi^* = 0 \end{aligned} \tag{2}$$

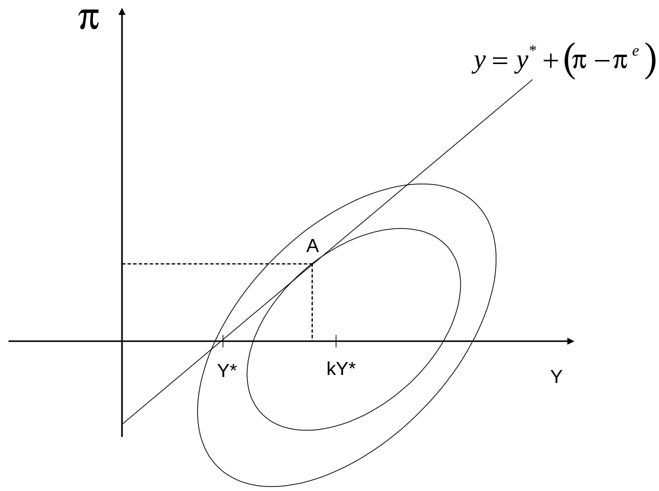

Figure 1: Barro-Gordon Model

where π^* is the inflation target, a is the importance of the inflation objective relative to the output one (greater a implies more weight to the inflation objective) and $k > 1$ means that the Bank has an output objective that is higher than the level of output (or employment) at which the economy would return normally. Assuming that the target inflation $\pi^* = 0$, for simplicity, (2) reduces to:

$$L = \frac{1}{2}\left[a\left(\pi\right)^2 + \left(y - ky^*\right)^2\right] \tag{3}$$

The combination of (3) and (1) can be represented using a simple graph in the space $y - \pi$, as shown in figure (1) where the point $\{y = k, \pi = 0\}$ represents the "bliss point" - the point that maximises the social welfare (minimises the social losses). Because this point is not on the Philips curve relationship, it cannot be achieved and thus the Bank has to decide the combination of output and inflation that is the closest to the bliss point on the Phillips curve (point A in the picture). Mathematically this point can be determined by substituting (1) in (3), deriving the resulting equation with respect to π and solving the resulting first order condition. Formally

$$L = \frac{1}{2}\left[a\left(\pi\right)^2 + \left(\left(1-k\right)y^* + b\left(\pi - \pi^e\right)\right)^2\right] \tag{4}$$

taking the derivative of (4) with respect to π we have the first order condition for an optimum, which is:

$$\frac{\partial L}{\partial \pi} = a\pi + b\left(\left(1-k\right)y^{*} + b\left(\pi - \pi^{e}\right)\right) = 0 \tag{5}$$

The first order condition can be solved for π to find the optimal level of inflation given the expectations of the private sector π^{e}

$$\pi = \frac{b}{a+b^{2}}\left(b\pi^{e} + \left(k-1\right)y^{*}\right) \tag{6}$$

if, like in the picture, $\pi^{e} = 0$, than we obtain the equlibrium level of inflation and output substituing (6) in (1) which gives the following values for inflation

$$\pi^{f} = \frac{b}{a+b^{2}}\left(\left(k-1\right)y^{*}\right) \tag{7}$$

and output

$$y^{f} = y^{*} + \frac{b^{2}}{a+b^{2}}\left(\left(k-1\right)y^{*}\right) = \frac{a+kb^{2}}{a+b^{2}}y^{*} \tag{8}$$

Inserting this equlibrium values in the loss function (2), we obtain the level of social welfare that this policy can achieve

$$\begin{aligned} L^{f} &= a\left[\frac{b}{a+b^{2}}\left(\left(k-1\right)y^{*}\right)\right]^{2} + \left[\frac{a\left(1-k\right)}{a+b^{2}}y^{*}\right]^{2} \\ &= \left(k-1\right)^{2}y^{*2}\left[\frac{ab^{2}+a^{2}}{\left(a+b^{2}\right)^{2}}\right] \\ &= \left(k-1\right)^{2}y^{*2}\frac{a}{\left(a+b^{2}\right)} \end{aligned} \tag{9}$$

Notice that this solution implies a level of inflation that is higher than the one expected by the private sector. If the bank just followed the promise of maintaining zero inflation tough, the social welfare obtained would have been worse than the one in (9), as you can see below:

$$\begin{aligned} \pi^{e} &= \pi = 0 \\ y^{d} &= y^{*} \\ L^{d} &= \left[\left(k-1\right)y^{*}\right]^{2} \end{aligned}$$

It is clear that $\left[\left(k-1\right)y^{*}\right]^{2} > \left(k-1\right)^{2}y^{*2}\frac{a}{\left(a+b^{2}\right)}$ and thus a policy of commitment to zero inflation would have implied higher social losses. But this result (corrisponding

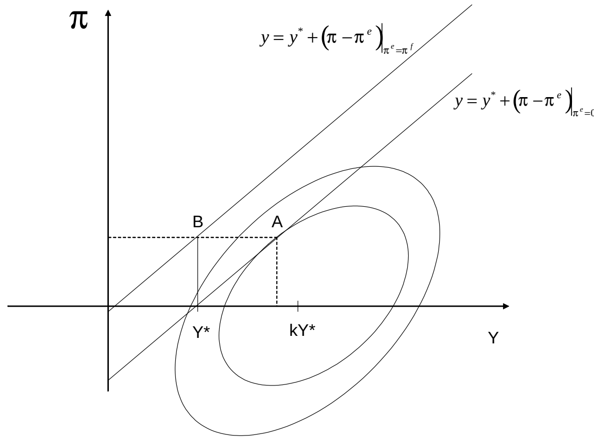

Figure 2: Barro Gordon with Rational Expectations

to point A in figure 1) is possible only becuase private sector expectations are actually wrong, i.e. the private sector is fooled by Central Bank behaviour. But the public can anticipate Central Bank incentives and therefore they can adjust their expectations $\pi^e = \pi^f$, producing an output level by (1) equal to the long term level of output $y = y^*$, lower than the desired level ky^*.

In figure (1) this means that the Phillips curve moves up (as shown in figure 2) and the resulting equilibrium (B) will be more socially expensive than the one with zero inflation.

Formally, substituting the expected inflation in (6) we obtain the equlibrium level of inflation and output determined by the discretionary policy of the Central Bank combined with the response of the private sector

$$\pi^c = \frac{b}{a + b^2} \left(b\pi^c + (k-1)\, y^* \right)$$

solving for π^c we have

$$\begin{aligned} \pi^c &= \frac{b}{a} \left((k-1)\, y^* \right) \\ y^c &= y^* \end{aligned}$$

which gives the losses

$$
\begin{aligned}
L^c &= a\left[\frac{b}{a}\left(\left(k-1\right)y^*\right)\right]^2 + \left[\left(1-k\right)y^*\right]^2 \\
&= \left(k-1\right)^2 y^{*2}\left(\frac{a+b^2}{a}\right)
\end{aligned}
$$

Notice that

$$
\left(k-1\right)^2 y^{*2}\left(\frac{a+b^2}{a}\right) > \left[\left(k-1\right)y^*\right]^2
$$

So the society would have been better off if the Bank had just followed a policy of zero inflation. If the bank had *Committed* to a policy of zero inflation and if the private sector had believed her, the sociaety would have been better off.

0.1 Effects of Uncertainty

The results in part change if we introduce uncertainty in the model. The timing of the game is now the following. Private sector form inflation expectations and fix wages. Shock to the economy are revealed and the Central Bank decides her monetary policy to stabilize the economy, togheter with achieving social output objectives. With Uncertainty the Phillips curve becomes:

$$
y = y^* + b\left(\pi - \pi^e\right) + \epsilon
$$

where ϵ is a random varialbe with mean equal to zero and variance equal to σ^2. Now the loss of the Central Bank can be written as:

$$
L = a\pi^2 + \left(\left(1-k\right)y^* + b\left(\pi - \pi^e\right) + \epsilon\right)^2
$$

The first order condition for an optimum will be equal to:

$$
\begin{aligned}
\frac{\partial L}{\partial \pi} &= 2a\pi + 2b\left(\left(1-k\right)y^* + b\left(\pi - \pi^e\right) + \epsilon\right) = 0 \\
\pi &= \frac{b}{a+b^2}\left(b\pi^e + \left(k-1\right)y^* - \epsilon\right)
\end{aligned}
$$

The private sector can anticipate the systematic part of policy but it cannot anticipate shocks. So private sector expectations will be

$$\pi^e = \frac{b}{a}(k-1)\,y^*$$

Substituting above from which we can derive the equlibrium level of inflation (after the shock) :

$$\begin{aligned}
\pi^d &= \frac{b}{a+b^2}\left[\frac{b^2}{a}(k-1)\,y^* + (k-1)\,y^* - \epsilon\right] \\
&= \frac{b}{a}\left((k-1)\,y^*\right) - \frac{b}{a+b^2}\epsilon
\end{aligned}$$

and the equlibrium level of output

$$\begin{aligned}
y^d &= y^* - \frac{b^2}{a+b^2}\epsilon + \epsilon \\
&= y^* + \frac{a}{a+b^2}\epsilon
\end{aligned}$$

Substituting these two values in the objective function we obtain the social losses of this policy

$$\begin{aligned}
E\left(L^d\right) &= a\left[\frac{b}{a}\left((k-1)\,y^*\right) - \frac{b}{a+b^2}\epsilon\right]^2 + \left[b\left(\pi^d - \pi^e\right) + (1-k)\,y^* + \epsilon\right]^2 \\
&= \frac{b^2}{a}(k-1)^2\,y^{*2} + \frac{ab^2}{(a+b^2)^2}\sigma^2 + (k-1)^2\,y^{*2} + \frac{a^2}{(a+b^2)^2}\sigma^2 \\
&= (k-1)^2\,y^{*2}\left(\frac{a+b^2}{a}\right) + \frac{a}{a+b^2}\sigma^2
\end{aligned}$$

In the case of a simple rule of inflation equal to zero, the expected losses would be instead :

$$E\left(L^r\right) = (k-1)^2\,y^{*2} + \sigma^2$$

Which policy is the best? Now it is not obvious anymore because the losses are made of two parts. The first element is the systematic losses due to output objective, that is certainly worse in the first case.

$$(k-1)^2\,y^{*2}\left(\frac{a+b^2}{a}\right) > (k-1)^2\,y^{*2}$$

But a second part of the losses is now represented by the level of fluctuations in the economy, and this part is certainly higher with the policy of zero inflation

$$\sigma^2 > \frac{a^2}{\left(a + b^2\right)^2}\sigma^2$$

With uncertainty it is not obvious that commiting to a rule improves economic performances. A policy of zero inflation is not efficient if the level of shocks affecting the economy is high. In this case a discretionary monetary policy that try to stabilize the economy might be more efficient from a social point oif view.